Hello everyone! My name is Osamu Kada. Today I'm going to prove Cayley-Hamilton Theorem. It states as follows.

fzF Theorem (Cayley-Hamilton)

Let A be an $n \times n$ matrix whose components are real numbers. (Instead of real numbers, we may take any commutative ring R).

And let $P_A(t) := \det(tI - A)$ be the characteristic polynomial of A, here I is the identity matrix. Then, by substituting A for t, we have that $P_A(A) = \mathrm{O}$, the zero matrix.

Here, for a polynomial $f(t) = \sum_{k=0}^{m} a_k t^k$, we define $f(A) = \sum_{k=0}^{m} a_k A^k$, and $A^0 = I$, the identity matrix.

For instance, by substituting A for t, $t^2 + 2$ turns out to $A^2 + 2I$, t turns out to A, number a turns out to aI, and 0 turns out to O, the zero matrix.

For $A = (a_{ij}) = \begin{pmatrix} a_{11} & \dots & a_{1n} \\ \vdots & & \vdots \\ a_{n1} & \dots & a_{nn} \end{pmatrix}$,

$$\begin{aligned} P_A(t) &= \left| \begin{pmatrix} t & & \\ & \ddots & \\ & & t \end{pmatrix} - \begin{pmatrix} a_{11} & \dots & a_{n1} \\ \vdots & & \vdots \\ a_{1n} & \dots & a_{nn} \end{pmatrix} \right| = \begin{vmatrix} t - a_{11} & \dots & -a_{n1} \\ \vdots & & \vdots \\ -a_{n1} & \dots & t - a_{nn} \end{vmatrix} \\ &= t^n - \operatorname{tr}(A) t^{n-1} + \dots + (-1)^n \det A, \text{ and} \\ P_A(A) &= A^n - \operatorname{tr}(A) A^{n-1} + \dots + (-1)^n \det(A) I. \end{aligned}$$

Consider more concretely the case when $n = 2$ and let $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$.

Then $\det P_A(t) = \det(tI - A) = t^2 - (a+d)t + ad - bc$, and by substituting A for t we have $\det P_A(A) = \det(AI - A) = \det \mathrm{O} = 0$? No, this is false.

Recall that by substituting A for t, t, a and 0 turns out to A, aI and O. respectively.

So, $\begin{pmatrix} t & 0 \\ 0 & t \end{pmatrix}$ turns out to $\begin{pmatrix} A & \mathrm{O} \\ \mathrm{O} & A \end{pmatrix}$, and $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ turns out to $\begin{pmatrix} aI & bI \\ cI & dI \end{pmatrix}$.

Here we are considering the matrices $\begin{pmatrix} A & \mathrm{O} \\ \mathrm{O} & A \end{pmatrix}$ and $\begin{pmatrix} aI & bI \\ cI & dI \end{pmatrix}$ as 2×2 matrix whose components are polynomials in the matrix A.

For example, note that the determinant of the matrix $\begin{pmatrix} I & \mathrm{O} \\ \mathrm{O} & I \end{pmatrix}$, considering it as 2×2 matrix whose components are polynomials in A, the determinant of the matrix is equal to $I^2 - \mathrm{O}^2 = I$.

But we can also consider it as 4×4 matrix whose components are real numbers, which is the usual one, and its determinant is equal to 1.

Similarly, the determinant of $\begin{pmatrix} A & \mathrm{O} \\ \mathrm{O} & A \end{pmatrix}$ is equall to $A^2 - \mathrm{O}^2 = A^2$ considering in $M_2(\mathbb{R}[A])$, and it is equall to $(\det A)^2$ considering in $M_4(\mathbb{R})$.

Now, back to the computation of $P_A(A)$.

$P_A(A) = A^2 - (a+d)A + (ad - bc)I$, and pulling out the A we have

$P_A(A) = \begin{pmatrix} -d & b \\ c & -a \end{pmatrix} A + (ad - bc)I$. The matrix $\begin{pmatrix} -d & b \\ c & -a \end{pmatrix}$ is equal to the negative of the adjugate matrix of A, so that

$P_A(A) = -\widetilde{A}A + (ad - bc)I = -\det(A)I + \det(A)I = \mathrm{O}$.

Now we prove Cayley-Hamilton theorem. Let
$B(t) := tI - A^T = \begin{pmatrix} t & & \\ & \ddots & \\ & & t \end{pmatrix} - \begin{pmatrix} a_{11} & \dots & a_{n1} \\ \vdots & & \vdots \\ a_{1n} & \dots & a_{nn} \end{pmatrix}$, here A^T is the transpose of A.
Then $B(A) = \begin{pmatrix} A & & \\ & \ddots & \\ & & A \end{pmatrix} - \begin{pmatrix} a_{11}I & \dots & a_{n1}I \\ \vdots & & \vdots \\ a_{1n}I & \dots & a_{nn}I \end{pmatrix} \in M_n(\mathbb{R}[A])$.
We are considering this matrix $B(A)$ as $n \times n$ matrix whose components are polynomials in the matrix A.
The ring of polynomials in A, $\mathbb{R}[A]$, is a commutative subring of the non-commutative ring of $n \times n$ matices, $M_n(\mathbb{R})$, we can apply theorems on matrix and determinant in $M_n(\mathbb{R}[A]$.

Let $\widetilde{B(A)}$ be the adjugate matrix of $B(A)$ considering in $M_n(\mathbb{R}[A])$, not in $M_{n^2}(\mathbb{R})$.
For instance, for $A \in M_2(\mathbb{R})$ and $C := \begin{pmatrix} A & \mathrm{O} \\ \mathrm{O} & \mathrm{O} \end{pmatrix} \in M_2(\mathbb{R}[A])$, the adjugate matrix of C considering in $M_2(\mathbb{R}[A])$ is $\begin{pmatrix} \mathrm{O} & \mathrm{O} \\ \mathrm{O} & A \end{pmatrix}$, but the adjugate matrix of C considering in $M_4(\mathbb{R})$ is $\begin{pmatrix} \mathrm{O} & \mathrm{O} \\ \mathrm{O} & \mathrm{O} \end{pmatrix}$.

Then we have that

$$\widetilde{B(A)}B(A) = \begin{pmatrix} \det B(A) & & \\ & \ddots & \\ & & \det B(A) \end{pmatrix} = \begin{pmatrix} \det P_A(A) & & \\ & \ddots & \\ & & \det P_A(A) \end{pmatrix}.$$

Since components of $B(A)$ are matrices, we can multiply $\begin{pmatrix} e_1 \\ \vdots \\ e_n \end{pmatrix}$ from the right, here $e_1, \dots, e_n$ is the canonical basis of $\mathbb{R}^n$. Then

$$B(A) \begin{pmatrix} e_1 \\ \vdots \\ e_n \end{pmatrix} = \begin{pmatrix} Ae_1 \\ \vdots \\ Ae_n \end{pmatrix} - \begin{pmatrix} a_{11}e_1 + \cdots + a_{n1}e_n \\ \vdots \\ a_{1n}e_1 + \cdots + a_{nn}e_n \end{pmatrix} = \begin{pmatrix} \vec{0} \\ \vdots \\ \vec{0} \end{pmatrix}.$$

Hence, we have that

$$\begin{pmatrix} \vec{0} \\ \vdots \\ \vec{0} \end{pmatrix} = \widetilde{B(A)}B(A) \begin{pmatrix} e_1 \\ \vdots \\ e_n \end{pmatrix} = \begin{pmatrix} P_A(A)e_1 \\ \vdots \\ P_A(A)e_n \end{pmatrix},$$

which implies that $P_A(A) = \mathrm{O}$.

Consider the following example. Let $A = \begin{pmatrix} & & -1 \\ & 0 & \\ 2 & & \end{pmatrix} = (2e_3,\ \vec{0},\ -e_1)$. Then

$$B(t) := tI - A^T = \begin{pmatrix} t & & \\ & t & \\ & & t \end{pmatrix} - \begin{pmatrix} & & 2 \\ & 0 & \\ -1 & & \end{pmatrix} = \begin{pmatrix} t & & -2 \\ & t & \\ 1 & & t \end{pmatrix},$$

$$B(A) = \begin{pmatrix} A & & \\ & A & \\ & & A \end{pmatrix} - \begin{pmatrix} & & 2I \\ & \mathrm{O} & \\ -I & & \end{pmatrix} = \begin{pmatrix} A & & -2I \\ & A & \\ I & & A \end{pmatrix},$$

$$\widetilde{B(A)} = \begin{pmatrix} A^2 & \mathrm{O} & 2A \\ \mathrm{O} & A^2+2I & \mathrm{O} \\ -A & \mathrm{O} & A^2 \end{pmatrix}, P_A(t) = \det B(t) = t^3 + 2t,$$

$$P_A(A) = \det \begin{pmatrix} A & & -2I \\ & A & \\ I & & A \end{pmatrix} = A^3 + 2A,$$

$$\widetilde{B(A)}B(A) = \begin{pmatrix} A^2 & \mathrm{O} & 2A \\ \mathrm{O} & A^2+2I & \mathrm{O} \\ -A & \mathrm{O} & A^2 \end{pmatrix} \begin{pmatrix} A & & -2I \\ & A & \\ I & & A \end{pmatrix} = \begin{pmatrix} A^3+2A & & \\ & A^3+2A & \\ & & A^3+2A \end{pmatrix}$$

$$B(A) \begin{pmatrix} e_1 \\ e_2 \\ e_3 \end{pmatrix} = \begin{pmatrix} A & & \\ & A & \\ & & A \end{pmatrix} \begin{pmatrix} e_1 \\ e_2 \\ e_3 \end{pmatrix} - \begin{pmatrix} & & 2I \\ & \mathrm{O} & \\ -I & & \end{pmatrix} \begin{pmatrix} e_1 \\ e_2 \\ e_3 \end{pmatrix}$$

$$= \begin{pmatrix} Ae_1 \\ Ae_2 \\ Ae_3 \end{pmatrix} - \begin{pmatrix} 2e_1 \\ 0 \\ -e_1 \end{pmatrix} = \begin{pmatrix} 2e_1 \\ 0 \\ -e_1 \end{pmatrix} - \begin{pmatrix} 2e_1 \\ 0 \\ -e_1 \end{pmatrix} = \begin{pmatrix} \vec{0} \\ \vec{0} \\ \vec{0} \end{pmatrix}.$$

So that $\begin{pmatrix} A^3+2A & & \\ & A^3+2A & \\ & & A^3+2A \end{pmatrix} \begin{pmatrix} e_1 \\ e_2 \\ e_3 \end{pmatrix} = \begin{pmatrix} (A^3+2A)e_1 \\ (A^3+2A)e_2 \\ (A^3+2A)e_3 \end{pmatrix} = \begin{pmatrix} \vec{0} \\ \vec{0} \\ \vec{0} \end{pmatrix}$,
which implies that $P_A(A) = A^3 + 2A = \mathrm{O}$.
Consider the case when $n = 2$ and $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. Then
$B(t) = tI - A^T = \begin{pmatrix} t & 0 \\ 0 & t \end{pmatrix} - \begin{pmatrix} a & c \\ b & d \end{pmatrix}$, and
$B(A) = \begin{pmatrix} A & \mathrm{O} \\ \mathrm{O} & A \end{pmatrix} - \begin{pmatrix} aI & cI \\ bI & dI \end{pmatrix} \in M_2(\mathbb{R}[A])$.

Since $B(A) \begin{pmatrix} e_1 \\ e_2 \end{pmatrix} = \begin{pmatrix} Ae_1 \\ Ae_2 \end{pmatrix} - \begin{pmatrix} ae_1 + ce_2 \\ be_1 + de_2 \end{pmatrix} = \begin{pmatrix} \begin{pmatrix} a \\ c \end{pmatrix} \\ \begin{pmatrix} b \\ d \end{pmatrix} \end{pmatrix} - \begin{pmatrix} \begin{pmatrix} a \\ c \end{pmatrix} \\ \begin{pmatrix} b \\ d \end{pmatrix} \end{pmatrix} = \begin{pmatrix} \begin{pmatrix} 0 \\ 0 \end{pmatrix} \\ \begin{pmatrix} 0 \\ 0 \end{pmatrix} \end{pmatrix}$,

we have that $\begin{pmatrix} P_A(A)e_1 \\ P_A(A)e_2 \end{pmatrix} = \widetilde{B(A)}B(A) \begin{pmatrix} e_1 \\ e_2 \end{pmatrix} = \begin{pmatrix} \begin{pmatrix} 0 \\ 0 \end{pmatrix} \\ \begin{pmatrix} 0 \\ 0 \end{pmatrix} \end{pmatrix}$, implying $P_A(A) = \mathrm{O}$.

$B(A)$ is not the zero matrix, but multiplying $\begin{pmatrix} e_1 \\ e_2 \\ e_3 \end{pmatrix}$ it turns out to zero.
This is sufficient to prove that $P_A(A) = \mathrm{O}$. the zero matrix.
And the mutiplication by its adjugate matrix is the zero matrix.